6. Optimal Cycles and Chaos

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6.1 Introduction

Optimal growth models have originally been developed in order to analyze the long-run implications of capital accumulation and technological progress. Later on, however, it has been noticed that essentially the same model structure can also be used to shed light on short-run phenomena like the business cycle. The most prominent outcome of this line of research are real-business-cycle (RBC) theories, which assume that business cycles are triggered by exogenous stochastic shocks and which analyze the mechanisms by which these shocks propagate through the economy. The literature surveyed in the present chapter, on the other hand, shows that optimal growth models can generate business cycles even in the absence of exogenous shocks. It is therefore appropriate to refer to these results as endogenous-business-cycle (EBC) theories. An important property common to both RBC and EBC theories is that the business cycles qualify as optimal programs. In other words, the solutions of both RBC and EBC models are Pareto-efficient. As far as the deterministic EBC models are concerned, the most important implication of this fact is that the standard assumptions of optimal growth theory do not rule out intrinsic instability of the economy, an instability that allows for periodic or even chaotic optimal programs.

Following Ramsey (1928), much of the earlier literature in optimal growth theory focused on equal treatment of generations over time, and therefore on the undiscounted case. The analysis of this class of models was brought to maturity in the papers of Gale (1967), McKenzie (1968), and Brock (1970). The treatment of the case in which future utilities were discounted was typically done in the one-sector neoclassical model, where the significant difference between the two cases was not revealed because of the one-to-one conversion of capital to consumption good implicit in its formulation. It was with the examples of Kurz (1968) and Sutherland (1970), in models which did not have this feature, that one recognized that discounting the future in general provided more limited intertemporal arbitrage opportunities; thus, the standard argument for smoothing out cyclical behavior was not valid in such frameworks.

Samuelson (1973) can be considered to provide definitive directions for research towards an understanding of such a phenomenon. On the one hand, he reported an example, due to Weitzman, which showed that cyclical optimal behavior was consistent with interior solutions to Ramsey-Euler equations and therefore would not disappear with assumptions which ruled out boundary solutions to optimal growth problems. On the other hand, he conjectured that, if the utility function was strictly concave, then cyclical optimal behavior of the Weitzman type would not arise, if the planner was sufficiently patient. The second idea was formalized in terms of turnpike theorems for low discount rates in a *Journal of Economic Theory* symposium of 1976, and led to a literature which is comprehensively surveyed in McKenzie (1986). The first idea led Benhabib and Nishimura (1985) to initiate a systematic investigation of the sources of optimal cycles and this, in turn, led to the literature that is surveyed in this chapter.

Section 6.2 sets the stage for our survey by presenting background material on dynamical systems and optimal growth models. Sections 6.3 and 6.4 form the main part of the chapter. In section 6.3 we study the optimality of periodic cycles. Although periodic optimal growth paths cannot be interpreted as realistic business cycles, the characterization of the conditions under which periodic cycles are optimal allows important insights into the mechanisms that can generate non-monotonic optimal growth paths. Section 6.4 then turns to chaotic optimal growth paths. These solutions resemble actual business cycles more closely than periodic ones, but it is somewhat harder to characterize the mechanisms by which they are generated.

6.2 Basic Definitions and Results

This section presents some background material that is necessary to state the main results on optimal cycles and chaos. First we discuss a number of concepts and results that are related to cyclical and chaotic behavior of dynamical systems. Then we formulate the reduced form optimal growth model and show that it encompasses, among other models, a discrete-time version of the two-sector model of Uzawa (1964).

6.2.1 Dynamical Systems

Let X be a non-empty set and let h be a map from X to X. The pair (X, h) is called a *dynamical system*. We refer to X as the *state space* and to h as the *law of motion* of the dynamical system. Thus, if $x_t \in X$ is the state of the system in time period t (where $t = 0, 1, 2, \ldots$), then $x_{t+1} = h(x_t) \in X$ is the state of the system in time period $t + 1$. We write $h^{(0)}(x) = x$ and, for any integer $t \geq 1$, $h^{(t)}(x) = h(h^{(t-1)}(x))$. If $x \in X$, the sequence $\tau(x) = (h^{(t)}(x))_{t=0}^{\infty}$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\omega(x) = \{y \,|\, y = h^{(t)}(x) \text{ for some } t \geq 0\}$.

A point $x \in X$ is a *fixed point* of the dynamical system (X, h), if $h(x) = x$. A point $x \in X$ is called a *periodic point* of (X, h), if there is $p \geq 1$ such that $h^{(p)}(x) = x$. The smallest such p is called the *period* of x. In particular, if $x \in X$ is a fixed point of (X, h), then it is periodic with period 1.

Throughout this chapter we assume that X is a non-empty and compact interval on the real line $\mathbb{R}$. In this case, it makes sense to describe the asymptotic behavior of a trajectory from x by its *limit set*, which is defined as the set of all limit points of $\tau(x)$. The limit set of x is denoted by $\omega_+(x)$. Note that, if $\hat{x} \in X$ is a periodic point, then $\omega_+(h^{(t)}(\hat{x})) = \omega(\hat{x})$ for every $t = 0, 1, 2, \ldots$. A periodic point $\hat{x}$ is said to be *locally stable*, if there is an open interval $I \subseteq X$ containing $\hat{x}$ such that $\omega_+(x) = \omega(\hat{x})$ for all $x \in I$. In this case we also say that the periodic orbit $\omega(\hat{x})$ is locally stable. If h is continuously differentiable on X and $\hat{x}$ is a periodic point of period p, then a sufficient condition for $\hat{x}$ to be locally stable is that $\left|Dh^{(p)}(\hat{x})\right| < 1$. If $\left|Dh^{(p)}(\hat{x})\right| > 1$, then $\hat{x}$ is not locally stable. A periodic point $\hat{x}$ is said to be *globally stable* (*almost globally stable*), if $\omega_+(x) = \omega(\hat{x})$ holds for all (almost all) initial points $x \in X$.

Suppose that the law of motion h is a non-decreasing function. Obviously, this implies that the trajectory $\tau(x)$ is a monotonic sequence for every $x \in X$. Because X is compact, this sequence must have a unique limit point. It follows therefore for every $x \in X$ that the limit set $\omega_+(x)$ is a singleton. Any form of non-monotonic behavior such as periodic orbits with a period $p \geq 2$ is therefore ruled out when h is non-decreasing. Now suppose that h is non-increasing. This implies that the second iterate $h^{(2)}$ is non-decreasing. Consequently, every limit set of the dynamical system $(X, h^{(2)})$ is a singleton and it follows that every limit set of the original system (X, h) consists of at most two points. A dynamical system (X, h) with a non-increasing law of motion can therefore have periodic points of period 2 but it cannot have periodic points of any period $p > 2$.

Consider the following complete order on the positive integers:

$$3 \prec 5 \prec 7 \prec \ldots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \ldots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec$$
$$\ldots \prec 2^n \cdot 3 \prec 2^n \cdot 5 \prec 2^n \cdot 7 \prec \ldots \prec 2^n \prec \ldots \prec 2^2 \prec 2 \prec 1.$$

This order is called the *Sarkovskii order*. Sarkovskii (1964) has proved that a dynamical system (X, h), where h is a continuous function, has the following property: if there exists a periodic point of period p and if $p \prec q$, then there exists also a periodic point of period q.

When we say that a dynamical system is history independent, we wish to convey the observation that the long-run (asymptotic) behavior of a typical trajectory is independent of the initial state. Formally, let (X, h) be a dynamical system and let λ be the Lebesgue measure on X. The dynamical system (X, h) is *history independent*, if there exists a subset E of X such that, for λ-almost every $x \in X$, the limit set of x satisfies $\omega_+(x) = E$. The dynamical system (X, h) is *history dependent*, if it is not history independent.

We will often be concerned with a family of dynamical systems, where the members of the family are indexed by a parameter. Formally, let us denote the parameter by $\delta \in D$, where D is taken to be a non-empty interval in $\mathbb{R}$. A *family of dynamical systems* will then be denoted by (X, h_δ), where h_δ maps X to X for each $\delta \in D$. Suppose the dynamical system (X, h_δ) is history independent for every $\delta \in D$. Then, for each $\delta \in D$, we can find a set $E(\delta)$ such that the following is true for Lebesgue almost every x in X: the limit set of the trajectory from x generated by the dynamical system (X, h_δ) is equal to $E(\delta)$. A *bifurcation map* is the correspondence which associates to each $\delta \in D$ its history independent limit set $E(\delta) \subseteq X$. A *bifurcation diagram* is a diagrammatic representation of the graph of the bifurcation map.

We now turn to chaotic behavior. There are several aspects of complicated dynamics that need to be taken into account. One of the most important ones is the so-called sensitivity with respect to initial conditions. A simple way to describe this property is as follows. The dynamical system (X, h) exhibits *geometric sensitivity*, if there exists a number $\gamma > 1$ such that the following is true: for every integer $\tau \geq 0$ there exists $\varepsilon > 0$ such that, for all $(x, y) \in X \times X$ with $\|x - y\| < \varepsilon$ and for all $t \in \{0, 1, 2, \ldots, \tau\}$, it holds that

$$\|h^{(t)}(x) - h^{(t)}(y)\| \geq \gamma^t \|x - y\|.$$

Sensitivity with respect to initial conditions is also captured by the concept of a scrambled set. Let us denote by P the set of all periodic points of the dynamical system (X, h). A subset S of the state space X is called a *scrambled set* for the dynamical system (X, h), if the following two conditions are satisfied. (i) For all pairs (x, y) satisfying $x \in S$ and $y \in S$ it holds that

$$\liminf_{t \to \infty} |h^{(t)}(x) - h^{(t)}(y)| = 0.$$

(ii) For all pairs (x, y) satisfying $x \in S$ and either $x \neq y \in S$ or $y \in P$ it holds that

$$\limsup_{t \to \infty} |h^{(t)}(x) - h^{(t)}(y)| > 0.$$

The dynamical system (X, h) is said to exhibit *topological chaos*, if there exists an uncountable scrambled set and a periodic point of a period that is not a power of 2. Note that due to Sarkovskii's theorem mentioned above, a dynamical system which has a continuous law of motion and which exhibits topological chaos must have infinitely many periodic points of different periods. A famous theorem by Li and Yorke (1975) states that the dynamical system (X, h) exhibits topological chaos, if h is continuous and if there exists a periodic point of period $p = 3$.

One problem with the definition of topological chaos is that the scrambled set can have Lebesgue measure 0. If this is the case, then the chaotic behavior may not be observable. A chaos definition that circumvents this problem is that of ergodic chaos. The dynamical system (X, h) exhibits *ergodic chaos*, if there exists an absolutely continuous (with respect to Lebesgue measure) probability measure μ on X which is invariant and ergodic under h. *Invariance* is the property that $\mu(\{x \in X \mid h(x) \in B\}) = \mu(B)$ holds for all measurable sets $B \subseteq X$. The invariant measure μ is *ergodic*, if, for every measurable set $B \subseteq X$ satisfying $\{x \in X \mid h(x) \in B\} = B$, it holds that $\mu(B) \in \{0, 1\}$.

Results by Lasota and Yorke (1973) and Li and Yorke (1978) show that the dynamical system (X, h) has geometric sensitivity and ergodic chaos, if there exists $\gamma > 1$ and a point $\tilde{x} \in X$ splitting X into two subintervals (recall that we assume X to be a compact interval on $\mathbb{R}$) such that (i) h is twice continuously differentiable on both subintervals, (ii) h is strictly increasing for $x < \tilde{x}$ and strictly decreasing for $x > \tilde{x}$, and (iii) $|h'(x)| \geq \gamma$ for all $x \in X \setminus \{\tilde{x}\}$.

6.2.2 Optimal Growth Models

We maintain the assumption that the state space X is a non-empty and compact interval on the real line $\mathbb{R}$. A *reduced form optimal growth model* on X is described by a triple (Ω, U, δ), where Ω is the *transition possibility set*, U is the (*reduced form*) *utility function*, and δ is the *discount factor*. The following assumptions on (Ω, U, δ) will be maintained throughout this chapter.

A.1: $\Omega \subseteq X \times X$ is non-empty, closed, and convex.

A.2: $U : \Omega \mapsto \mathbb{R}$ is a continuous and concave function.

A.3: $0 < \delta < 1$.

A *program* from $x \in X$ is a sequence $(x_t)_{t=0}^{\infty}$ satisfying $x_0 = x$ and $(x_t, x_{t+1}) \in \Omega$ for all $t \geq 0$. Let $(x_t)_{t=0}^{\infty}$ be a program from $x \in X$. It is called an *optimal program* from x, if

$$\sum_{t=0}^{\infty} \delta^t U(x_t, x_{t+1}) \geq \sum_{t=0}^{\infty} \delta^t U(y_t, y_{t+1})$$

holds for every program $(y_t)_{t=0}^{\infty}$ from x.

The issues of existence and uniqueness of optimal programs have been well studied; see, e.g., Stokey and Lucas (1989) or Mitra (2000). Under assumptions A.1-A.3 there exists an optimal program from every $x \in X$. Thus, one can define the *value function* $V : X \mapsto \mathbb{R}$ by

$$V(x) = \sum_{t=0}^{\infty} \delta^t U(x_t, x_{t+1}),$$

where $(x_t)_{t=0}^{\infty}$ is an optimal program from x. The value function V is concave and continuous on X. Moreover, for all $x \in X$, the Bellman equation

$$V(x) = \max \{U(x,z) + \delta V(z) \,|\, (x,z) \in \Omega\}$$

holds.

For each $x \in X$, we denote by $h(x)$ the set of all $z \in X$ which maximize $U(x,z) + \delta V(z)$ over all $z \in X$ satisfying $(x,z) \in \Omega$. That is, for each $x \in X$,

$$h(x) = \operatorname{argmax} \{U(x,z) + \delta V(z) \,|\, (x,z) \in \Omega\}\,.$$

A program from $x \in X$ is an optimal program, if and only if $V(x_t) = U(x_t, x_{t+1}) + \delta V(x_{t+1})$ for $t \geq 0$, that is, if and only if $x_{t+1} \in h(x_t)$ holds for all $t \geq 0$. We call h the *optimal policy correspondence*.

Given an initial state $x \in X$, there may be more than one optimal program from x. If, for every $x \in X$, there is a unique optimal program from x, then it follows that the optimal policy correspondence h is a (single-valued) function. It can also be shown that this function is continuous on X. A simple condition that ensures the uniqueness of optimal programs is the strict concavity of the utility function U with respect to its second argument. Whenever the optimal policy correspondence is a single-valued function, we shall refer to it as the *optimal policy function*.

Reduced form optimal growth models arise in many different contexts. For the purpose of the present chapter, the *two-sector optimal growth model* introduced by Uzawa (1964) is the most relevant one. The state variable x_t of this model describes the aggregate capital stock available in the economy at the beginning of period t. There are two production sectors, one producing a consumption good and the other a capital good. Each sector uses the capital good and labor as inputs. The capital good cannot be consumed and depreciates at the rate d, where $d \in [0,1]$. The labor supply is assumed to be constant and equal to 1. Denoting the production functions in the consumption good sector and the capital good sector by $c = F_c(x_c, \ell_c)$ and $y = F_x(x_x, \ell_x)$, respectively, and the utility function by $u(c)$, the two-sector model can be formulated as follows.

$$\begin{aligned}
\text{Maximize} \quad & \sum_{t=0}^{\infty} \delta^t u(c_t) \\
\text{subject to} \quad & c_t \leq F_c(x_{c,t}, \ell_{c,t}) \\
& (1-d)x_t \leq x_{t+1} \leq F_x(x_{x,t}, \ell_{x,t}) + (1-d)x_t \\
& x_{c,t} + x_{x,t} \leq x_t \\
& \ell_{c,t} + \ell_{x,t} \leq 1.
\end{aligned}$$

In order to convert the two-sector optimal growth model into its reduced form, we first determine the transition possibility set Ω. To this end note that the set of all capital stocks that can be reached from the state x within one period is given by $\{z \,|\, (1-d)x \leq z \leq F_x(x,1) + (1-d)x\}$. If $d > 0$ and if the production function F_x is increasing and concave and satisfies the Inada conditions, then $F_x(x,1) + (1-d)x$ is an increasing and concave function of x. The slope of this function is strictly greater than 1 for small x and strictly smaller than 1 for large x. These properties imply that there exists a unique value $\bar{x} > 0$ satisfying $\bar{x} = F_x(\bar{x},1) + (1-d)\bar{x}$. The following properties hold for any pair (x,z), where z is a capital stock that can be reached within one period from x: if $x \leq \bar{x}$ then $z \leq \bar{x}$, and if $x > \bar{x}$ then $z < x$. In other words, $\bar{x}$ is the maximal sustainable capital stock. For this reason, it is justified to restrict attention to the compact state space $X = [0, \bar{x}]$. The transition possibility set Ω is therefore given by

$$\Omega = \{(x,z) \,|\, 0 \leq x \leq \bar{x}\,,\ (1-d)x \leq z \leq F_x(x,1) + (1-d)x\}$$

and the reduced form utility function is given by $U(x,z) = u(T(x,z))$, where

$$\begin{array}{rcl} T(x,z) & = & \max F_c(x_c, \ell_c) \\ \text{subject to} & & z \leq F_x(x_x, \ell_x) + (1-d)x \\ & & x_c + x_x \leq x \\ & & \ell_c + \ell_x \leq 1. \end{array}$$

The function T describes, for any given $x \in X$, the trade-off between consumption and capital production.

6.3 Optimal Cycles

Consider a reduced form optimal growth model (Ω, U, δ) on the state space X. Turnpike theory, as developed for example by Scheinkman (1976) or McKenzie (1983, 1986), shows that an optimal program for this model must be convergent provided that certain regularity assumptions are satisfied and the discount factor δ is sufficiently close to 1. In other words, given X, Ω, and U, a sufficiently large time-preference factor δ rules out cyclical or more complicated optimal programs. Whether complicated dynamic patterns can be optimal for small values of the discount factor, however, is left open by turnpike theory and has been the focus of intensive research in the 1980s and 1990s. The present section surveys some important contributions to this literature.

We start by summarizing a few results regarding the monotonicity properties of the optimal policy function. These results provide simple conditions for the non-existence of optimal cycles. The main part of this section, however, is concerned with the question of how long-run optimal behavior is affected by changes in the rate at which the future is discounted. In particular we will observe period-2 cycles being born and changing their amplitude as the discount factor δ falls.

The class of examples that we study in detail allow for period-2 cycles but no more complicated behavior than that, and they indicate an interesting feature about the transition from global asymptotic stability of a (unique) fixed point at high discount factors to global asymptotic stability of cycles at lower discount factors. The family of examples in subsection 6.3.3 constitute variations of the example of Weitzman, as discussed in Samuelson (1973). Weitzman's example has been widely discussed in the literature; see, for example, Scheinkman (1976), McKenzie (1983), Benhabib and Nishimura (1985), and Mitra and Nishimura (2001). For this family of examples there is a critical discount factor, $\hat{\delta}$, such that the following is true. For $\delta > \hat{\delta}$, all optimal programs converge to the unique fixed point $\hat{x}_\delta$ and, for $\delta < \hat{\delta}$, almost all optimal programs converge to a period-2 cycle. The bifurcation diagram also indicates that the amplitude of the period-2 cycle is monotonic in the discount factor.

The class of examples in subsection 6.3.4 constitute variations of the example presented by Sutherland (1970). This example has been discussed in Cass and Shell (1976), Benhabib and Nishimura (1985), and Mitra and Nishimura (2001). For this family of examples, too, there is a critical discount factor, $\hat{\delta}$, such that all optimal programs converge to the unique fixed point, if $\delta > \hat{\delta}$, and that almost all optimal programs converge to a period-2 cycle for $\delta < \hat{\delta}$. The range of discount factors can be further subdivided according to whether the period-2 cycle hits one boundary of the state-space or the other (or both). The global bifurcation diagram reveals that the amplitude of the period-2 cycle is *not* monotonic in the discount factor.

6.3.1 Monotonic Policy Functions

We have seen in subsection 6.2.1 that a dynamical system with a non-decreasing law of motion cannot generate cycles, and that a dynamical system with a non-increasing law of motion can generate cycles of period 2 but no cycles of any period $p > 2$. A first step towards the analysis of optimal cycles is therefore the investigation of the conditions that generate a non-decreasing or a non-increasing optimal policy function, respectively. Very general results in this respect can be derived by the use of the lattice theoretic concepts of super- and submodularity; see, e.g., Topkis (1978) or Ross (1983). The function $U : \Omega \mapsto \mathbb{R}$ is said to be *supermodular* if, for any two pairs $(x, z) \in \Omega$ and $(x', z') \in \Omega$, the following is true: if $x < x'$, $z < z'$, $(x, z') \in \Omega$, and $(x', z) \in \Omega$, then it holds that $U(x, z) + U(x', z') \geq U(x, z') + U(x', z)$. The function U is *submodular* if the inequality holds in reverse, that is, if $U(x, z) + U(x', z') \leq U(x, z') + U(x', z)$. The following theorem is a variant of a result stated in Amir (1996); see also Mitra (2000).

Theorem 6.3.1. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1-A.3 and assume that its optimal programs are described by the optimal policy function h. If U is supermodular (submodular), then h is non-decreasing (non-increasing) locally around every point x satisfying $(x, h(x)) \in \operatorname{int} \Omega$.*

Proof. We present the proof for the case where U is submodular; the case of a supermodular utility function can be dealt with analogously. Let $x \in X$

be a state such that $(x, h(x)) \in \operatorname{int} \Omega$. Since the optimal policy function is continuous, it follows that for every x' sufficiently close to x, the pairs $(x, h(x'))$, $(x', h(x))$, and $(x', h(x'))$ are also contained in Ω. Let us define $z = h(x)$ and $z' = h(x')$. Without loss of generality we may assume $x < x'$. We need to show that $z \geq z'$. Suppose to the contrary that $z < z'$. Because optimal programs are described by an optimal policy function, the maximizer of the right-hand side of the Bellman equation must be unique. Since $z \neq z'$ it follows therefore that $U(x, z) + \delta V(z) > U(x, z') + \delta V(z')$ and $U(x', z') + \delta V(z') > U(x', z) + \delta V(z)$, where V is the value function. Adding these inequalities it follows that $U(x, z) + U(x', z') > U(x, z') + U(x', z)$. This is a contradiction to the assumed submodularity of U, which completes the proof.

Theorem 6.3.1 shows that any interior section of the graph of the optimal policy function of a model with a supermodular (submodular) utility function is a non-decreasing (non-increasing) curve. If we have additional information about the transition possibility set, then it is possible to establish global monotonicity properties of the optimal policy function. This is shown in the following corollary which uses the definitions $\psi(x) = \min\{z \,|\, (x, z) \in \Omega\}$ and $\phi(x) = \max\{z \,|\, (x, z) \in \Omega\}$. Note that assumption A.1 implies that ψ and ϕ are continuous functions on X.

Corollary 6.3.1. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1-A.3 and assume that its optimal programs are described by the optimal policy function h.*
(i) If ψ and ϕ are non-decreasing functions and if U is supermodular, then it follows that h is non-decreasing on X.
(ii) If ψ and ϕ are non-increasing functions and if U is submodular, then it follows that h is non-increasing on X.

An important case in which the functions ψ and ϕ are non-decreasing is the two-sector model discussed in subsection 6.2.2. A simple example in which the functions ψ and ϕ are non-increasing is given by $\Omega = X \times X$.

If U is a twice continuously differentiable function, then supermodularity (submodularity) follows from $U_{12}(x, z) > 0$ ($U_{12}(x, z) < 0$); see, e.g., Ross (1983). This observation can be used to prove the following result due to Benhabib and Nishimura (1985). In order to formulate it, one needs to impose a smoothness condition on the utility function.

A.4: The utility function U is twice continuously differentiable on the interior of Ω with second-order partial derivatives U_{11}, U_{12}, and U_{22}. Moreover, it holds that $U_{11}(x, z) < 0$, $U_{22}(x, z) < 0$, and $U_{11}(x, z)U_{22}(x, z) - U_{12}(x, z)^2 \geq 0$ for all (x, z) in the interior of Ω.

Note that assumption A.4 implies strict concavity of the utility function with respect to its second argument which, in turn, implies that an optimal program from any initial state $x \in X$ is unique. In other words, optimal programs can be described by an optimal policy function.

Theorem 6.3.2. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1-A.4 and let h be its optimal policy function.*
(i) If $(x, h(x)) \in \operatorname{int} \Omega$ and $U_{12}(x, h(x)) > 0$ ($U_{12}(x, h(x)) < 0$), then it follows that h is non-decreasing (non-increasing) locally at x.
(ii) If $(x, h(x)) \in \operatorname{int} \Omega$, $(h(x), h^{(2)}(x)) \in \operatorname{int} \Omega$, and $U_{12}(x, h(x)) > 0$ ($U_{12}(x, h(x)) < 0$), then it follows that h is strictly increasing (strictly decreasing) locally at x.

Proof. Part (i) follows by the same argument that has been used in the proof of theorem 6.3.1 because $U_{12}(x, h(x)) > 0$ ($U_{12}(x, h(x)) < 0$) implies supermodularity (submodularity) of U locally around $(x, h(x))$. To prove part (ii), we simply have to show that $x \neq x'$ implies $h(x) \neq h(x')$. Suppose to the contrary that $h(x) = h(x')$, where x' is sufficiently close to x such that $(x', h(x))$ is in the interior of Ω. It follows that both $(x, h(x), h^{(2)}(x), \ldots)$ and $(x', h(x), h^{(2)}(x), \ldots)$ are optimal paths and, since the three points $(x, h(x))$, $(x', h(x))$, and $(h(x), h^{(2)}(x))$ are in the interior of Ω, the Euler equation $U_2(y, h(x)) + \delta U_1(h(x), h^{(2)}(x)) = 0$ must hold for $y \in \{x, x'\}$. Obviously, this is not possible if $x' \neq x$, x' is sufficiently close to x, and $U_{12}(x, h(x)) \neq 0$.

6.3.2 The Role of Discounting

We now turn to the question of how the existence or non-existence of a period-2 cycle depends on the size of the discount factor δ. This question has been thoroughly investigated by Mitra and Nishimura (2001), and the rest of this section draws heavily from their paper. In order to be able to develop a precise characterization, Mitra and Nishimura (2001) restrict the class of optimal growth models by a number of assumptions. These assumptions ensure that the dynamical system (X, h) is history independent, that optimal programs converge either to fixed points or to period-2 cycles, and that the asymptotic behavior of optimal programs depends in a simple way on the discount factor. We summarize their arguments in the present subsection. Subsections 6.3.3 and 6.3.4 will then illustrate the application of these ideas by means of two important classes of examples.

Mitra and Nishimura (2001) postulate the following strengthened version of assumption A.1.

A.1$^+$: It holds that $X = [0, 1]$ and $\Omega = X \times X$.

In addition to A.1$^+$ and A.2-A.4, they impose strict monotonicity and submodularity of the utility function.

A.5: For all (x, z) in the interior of Ω it holds that $U_1(x, z) > 0$, $U_2(x, z) < 0$, and $U_{12}(x, z) < 0$.

The monotonicity part of assumption A.5 is standard, submodularity is assumed to ensure that optimal programs can exhibit period-2 cycles but no more complicated behavior. The following result is a straightforward consequence of the results stated in the previous subsection.

Lemma 6.3.1. *Let (Ω, U, δ) be an optimal growth model satisfying assumptions A.1$^+$ and A.2-A.5. There exists an optimal policy function h. The optimal policy function is continuous and non-increasing on X. There exists a unique fixed point of the dynamical system (X, h). All optimal programs converge either to the fixed point or to a period-2 cycle.*

At this point it is important to emphasize the fact that the optimal policy function h and, therefore, its fixed point depend on the discount factor. Henceforth, we consider (Ω, U) as fixed and treat δ as a parameter varying between 0 and 1. In order to ensure that the fixed point of (X, h) is in the interior of the state space, Mitra and Nishimura (2001) postulate the following assumption.

A.6: Let the function $\pi : (0,1) \mapsto \mathbb{R}$ be defined by $\pi(x) = -U_2(x,x)/U_1(x,x)$. It holds that $\lim_{x\to 0} \pi(x) = 0$ and $\lim_{x\to 1} \pi(x) > 1$.

It is now possible to prove the following result.

Lemma 6.3.2. *Let Ω and U be given such that assumptions A.1$^+$, A.2, and A.4-A.6 are satisfied. For all $\delta \in (0,1)$ let $h_\delta : X \mapsto X$ be the optimal policy function of (Ω, U, δ) and let $\hat{x}_\delta$ be the unique fixed point of (X, h_δ).*
(i) The inequality $0 < \hat{x}_\delta < 1$ holds for all $\delta \in (0,1)$.
(ii) The fixed point $\hat{x}_\delta$ is continuously differentiable and strictly increasing with respect to δ and it holds that $\lim_{\delta\to 0} \hat{x}_\delta = 0$ and $\hat{x}_1 := \lim_{\delta\to 1} \hat{x}_\delta \in (0,1)$.

The general strategy of Mitra and Nishimura (2001) proceeds now as follows. First, an auxiliary problem is formulated which involves optimization over two periods only and in which the terminal state is restricted to be the same as the initial state. The unique optimal policy function of that problem is denoted by f_δ. Due to the construction of the auxiliary problem, it follows that the fixed point of (X, h_δ) coincides with the fixed point of (X, f_δ) and that interior period-2 cycles of (X, h_δ) coincide with those generated by (X, f_δ). Furthermore, the value of the derivative of f_δ at the fixed point $\hat{x}_\delta$ gives information about the eigenvalues of the Euler equation of (Ω, U, δ) at $\hat{x}_\delta$ which, in turn, determine the local stability (or instability) of $\hat{x}_\delta$ as a fixed point of (X, h_δ). A condition is then imposed on f_δ which ensures that local stability of $\hat{x}_\delta$ implies almost global asymptotic stability of $\hat{x}_\delta$, and that instability of $\hat{x}_\delta$ implies almost global stability of a period-2 cycle of (X, h_δ). Finally, Mitra and Nishimura (2001) impose another condition which ensures that there is only a single switching from local stability of $\hat{x}_\delta$ to instability (and no switch back to local stability) as the discount factor changes from 1 to 0. We shall now briefly explain the most important details of this strategy.

For any given $x \in X$, consider the following auxiliary optimization problem:

$$\begin{array}{ll} \text{Maximize} & U(x,z) + \delta U(z,x) \\ \text{subject to} & z \in X. \end{array}$$

Given our assumptions, this problem has a unique optimal solution which we denote by $f_\delta(x)$. If $(x, f_\delta(x)) \in \operatorname{int} \Omega$, then we have the first-order condition $U_2(x, f_\delta(x)) + \delta U_1(f_\delta(x), x) = 0$. Since $U_{22}(x, f_\delta(x)) + \delta U_{11}(f_\delta(x), x) < 0$, one can apply the implicit function theorem to conclude that f_δ is differentiable at x and that

$$f'_\delta(x) = -\frac{U_{12}(x, f_\delta(x)) + \delta U_{12}(f_\delta(x), x)}{U_{22}(x, f_\delta(x)) + \delta U_{11}(f_\delta(x), x)}. \tag{6.1}$$

Let us denote the second iterate of f_δ by F_δ, that is $F_\delta = f_\delta^{(2)}$. The following so-called history independence condition plays a crucial role in Mitra and Nishimura (2001).

A.7: If a, b, and c are fixed points of (X, F_δ) satisfying $a < b < c$, then it follows that $F'_\delta(b) > 1$.

From (6.1) one can easily see that $f'_\delta(x) < 0$ holds whenever $(x, f_\delta(x)) \in \operatorname{int} \Omega$. The absolute value of $f'_\delta(\hat{x}_\delta)$ contains information about the local stability or instability of the fixed point $\hat{x}_\delta$ with respect to the dynamical system (X, h_δ). Assumption A.7 (history independence) allows one to link the local behavior of (X, h_δ) around $\hat{x}_\delta$ to global properties. The details are summarized in the following theorem.

Theorem 6.3.3. *Let (Ω, U, δ) be an optimal growth model satisfying A.1$^+$ and A.2-A.7.*
(i) If $-1 < f'_\delta(\hat{x}_\delta) < 0$, then $\hat{x}_\delta$ is a locally stable fixed point of (X, h_δ). For all $x \in (0, 1)$ it holds that the optimal program from x converges to $\hat{x}_\delta$.
(ii) If $f'_\delta(\hat{x}_\delta) < -1$, then $\hat{x}_\delta$ is an unstable fixed point of (X, h_δ). There exists a period-2 cycle of (X, h_δ) with orbit $\{x^*_\delta, z^*_\delta\}$ such that the following is true: for all $x \in (0, \hat{x}_\delta) \cup (\hat{x}_\delta, 1)$ it holds that the optimal program from x converges to this period-2 cycle, that is, $\omega_+(x) = \{x^*_\delta, z^*_\delta\}$.

The above theorem implies that, under the stated assumptions, the set

$$\{(X, h_\delta) \mid \delta \in (0, 1), f'_\delta(\hat{x}_\delta) \neq -1\}$$

is a family of history independent dynamical systems. It makes therefore sense to consider the bifurcation diagram of this family with δ as the bifurcation parameter. In order to construct this diagram, one needs to find out for which δ it holds that $-1 < f'_\delta(\hat{x}_\delta) < 0$ and for which δ it holds that $f'_\delta(\hat{x}_\delta) < -1$. To this end, first note that assumption A.4 implies that $\max\{|U_{11}(\hat{x}_1, \hat{x}_1)|, |U_{22}(\hat{x}_1, \hat{x}_1)|\} \geq |U_{12}(\hat{x}_1, \hat{x}_1)|$, where as before $\hat{x}_1 = \lim_{\delta \to 1} \hat{x}_\delta$. Consider the following slight strengthening of this condition.

A.8: It holds that $\max\{|U_{11}(\hat{x}_1, \hat{x}_1)|, |U_{22}(\hat{x}_1, \hat{x}_1)|\} > |U_{12}(\hat{x}_1, \hat{x}_1)|$.

Under A.4, A.5, and A.8 one has $U_{11}(\hat{x}_1, \hat{x}_1) + U_{22}(\hat{x}_1, \hat{x}_1) < 2U_{12}(\hat{x}_1, \hat{x}_1)$. Since $\hat{x}_\delta$ is continuous with respect to δ, it must therefore hold for all δ sufficiently close to 1 that

$$U_{11}(\hat{x}_\delta, \hat{x}_\delta) + U_{22}(\hat{x}_\delta, \hat{x}_\delta) < (1 + \delta)U_{12}(\hat{x}_\delta, \hat{x}_\delta). \tag{6.2}$$

Because of (6.1) this implies $-1 < f'_\delta(\hat{x}_\delta) < 0$. Thus, according to theorem 6.3.3, for δ sufficiently large, the unique fixed point $\hat{x}_\delta$ is almost globally stable. If the inequality in (6.2) is reversed, however, the fixed point loses its stability and a period-2 cycle becomes almost globally stable. A sufficient condition for there to be exactly one switch from stability of the fixed point to its instability as δ decreases from 1 to 0 is the following so-called unique switching condition from Mitra and Nishimura (2001).

A.9: The function $R : (0,1) \mapsto (0,\infty)$ defined by

$$R(x) = \frac{U_2(x,x)U_{11}(x,x) - U_1(x,x)U_{22}(x,x)}{[U_2(x,x) - U_1(x,x)]U_{12}(x,x)}$$

is strictly increasing and satisfies $0 < \lim_{x\to 0} R(x) < 1$.

Indeed Mitra and Nishimura (2001) prove the following result.

Theorem 6.3.4. *For each $\delta \in (0,1)$ let (Ω, U, δ) be an optimal growth model satisfying A.1$^+$, A.2, and A.4-A.7. Furthermore, assume that A.8 and A.9 are satisfied. Then there exists a unique critical discount factor $\hat{\delta} \in (0,1)$ satisfying $R(\hat{x}_{\hat{\delta}}) = 1$. The following properties are true:*
(i) If $\hat{\delta} < \delta < 1$, then the unique fixed point $\hat{x}_\delta$ is almost globally stable.
(ii) If $0 < \delta < \hat{\delta}$, then there exists a period-2 cycle which is almost globally stable.

6.3.3 Variations on Weitzman's Example

In this section we discuss the case in which the utility function U is given by

$$U(x,z) = x^\alpha (1-z)^\beta,$$

where α and β are positive parameters satisfying $\alpha + \beta \le 1$. Using the Euler equation, it is easy to verify that the unique fixed point of (X, h_δ) is given by $\hat{x}_\delta = \alpha\delta/(\alpha\delta + \beta)$. Furthermore, the function R from assumption A.9 is given by

$$R(x) = [1 - \alpha + (\alpha - \beta)x]/[\alpha - (\alpha - \beta)x]. \tag{6.3}$$

The special case in which $\alpha = \beta = 1/2$ is Weitzman's example (as reported in Samuelson (1973)). For this special case it is known that, for every $\delta \in (0,1)$, the optimal policy function is given by

$$h_\delta(x) = \delta^2(1-x)/[x + \delta^2(1-x)],$$

and that every $x \neq \hat{x}_\delta$ is a periodic point of period 2 of the dynamical system (X, h_δ). This implies in particular that the limit sets $\omega_+(x)$ and $\omega_+(z)$ of any two points $x \in X$ and $z \neq h_\delta(x)$ are different from each other. The dynamical system (X, h_δ) is therefore history dependent. It is also worth pointing out that, in Weitzman's example, the function R from (6.3) is constant and equal to 1, which shows that A.9 fails to be satisfied and underlines the degenerate nature of this example. Slight modifications of Weitzman's example, however, lead to history independent optimal policy functions and can be dealt with by the methods from the previous subsection.

First of all, it is easy to see that the function R from (6.3) satisfies $0 < \lim_{x\to 0} R(x) < 1$, if and only if $\alpha > 1/2$. Because of $\alpha+\beta \leq 1$, this implies that $\alpha > \beta$ and it follows that R is strictly increasing. Thus, in order for assumption A.9 to be satisfied, it is necessary and sufficient to have $\alpha > 1/2$. It can be shown that assumptions A.1$^+$ and A.2-A.8 hold also under this assumption. The function R attains the critical value 1 at the value $\hat{x}_{\hat{\delta}} = (2\alpha-1)/[2(\alpha-\beta)]$. The corresponding critical value of the discount factor is $\hat{\delta} = (2\alpha-1)\beta/[\alpha(1-2\beta)]$. Let us now distinguish between the two cases $\alpha+\beta = 1$ and $\alpha+\beta < 1$.

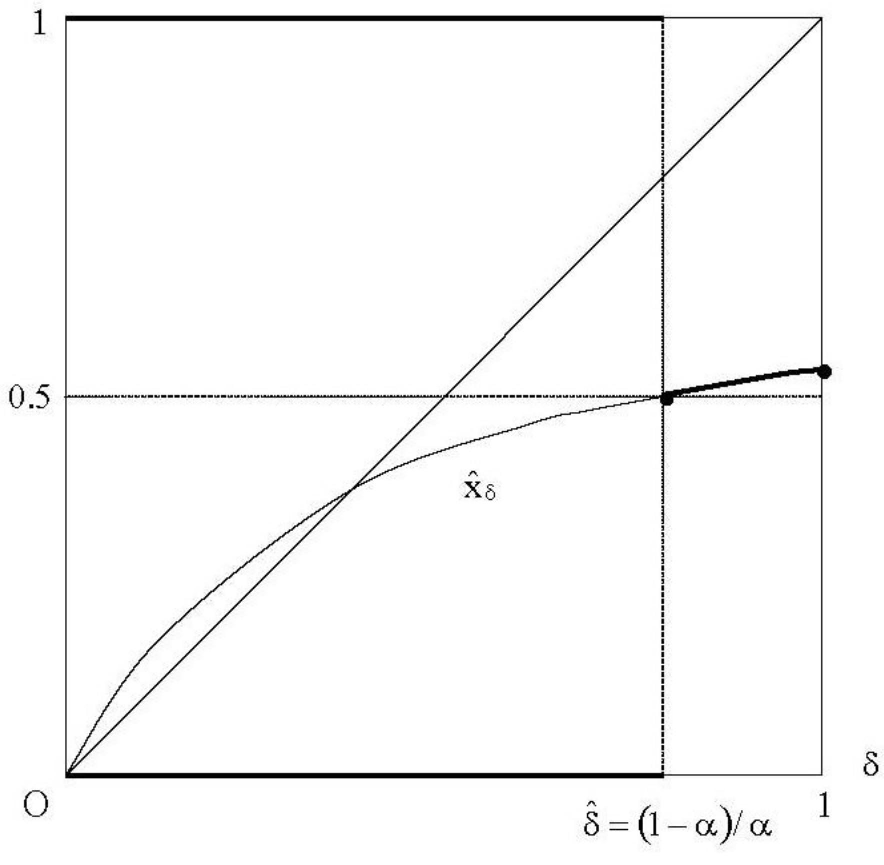

Fig. 6.1. The modified Weitzman example with $1/2 < \alpha = 1-\beta < 1$.

If $\alpha = 1-\beta > 1/2$, then we have $\hat{\delta} = (1-\alpha)/\alpha$ and $\hat{x}_{\hat{\delta}} = 1/2$. For $\delta \in (\hat{\delta}, 1)$, the fixed point $\hat{x}_\delta$ is globally stable. For $\delta \in (0, \hat{\delta})$, optimal programs from all initial stocks other than $\hat{x}_\delta$ converge to the period-2 boundary cycle with orbit

$\{0, 1\}$. At the bifurcation point $\delta = \hat{\delta} = (1 - \alpha)/\alpha$ one has neutral cycles, that is, starting from any initial stock $x \in X$, the period-2 cycle with orbit $\{x, 1-x\}$ is optimal; see figure 6.1.

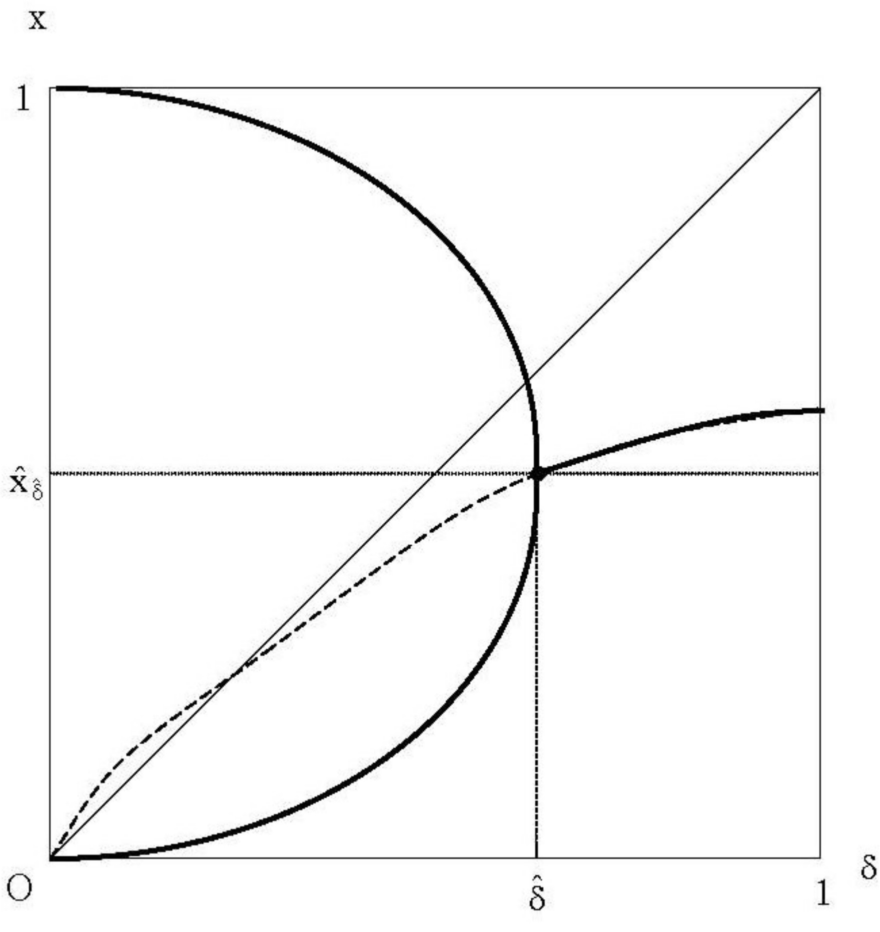

Fig. 6.2. The modified Weitzman example with $1/2 < \alpha < 1 - \beta < 1$.

Now consider the case where $\alpha + \beta < 1$. For $\delta \in (\hat{\delta}, 1)$, the fixed point $\hat{x}_\delta$ is globally stable. For $\delta \in (0, \hat{\delta})$, optimal programs from all initial states other than $\hat{x}_\delta$ converge to a unique period-2 interior cycle. This interior cycle has small amplitude for δ close to $\hat{\delta}$. The amplitude increases as δ falls and, as δ

converges to zero, the orbit of this period-2 cycle approaches $\{0, 1\}$. Thus, we obtain the standard bifurcation diagram of a flip bifurcation; see figure 6.2.

6.3.4 Variations on Sutherland's Example

We now consider the case in which the utility function U is given by

$$U(x, z) = -ax^2 - bxz - cz^2 + dx.$$

It is assumed that a, b, c, and d are positive real numbers satisfying $4ac > b^2$, $b > 2c$, and $2(a + b + c) > d > 2b(a - c)/(b - 2c)$. Note that these parameter restrictions imply that U is strictly concave and that $a > c$, $d > 2a > b$, and $a + c > b$. From the Euler equation one can see that the unique fixed point of (X, h_δ) is given by $\hat{x}_\delta = d\delta/[b + 2c + \delta(2a + b)]$ and that $\hat{x}_\delta \in (0, 1)$. The function R from assumption A.9 is given by

$$R(x) = [2cd + 2b(a - c)x]/[bd - 2b(a - c)x].$$

Given the above mentioned parameter restrictions, assumptions A.1$^+$ and A.2-A.9 are satisfied. The function R attains the critical value 1 at the value $\hat{x}_{\hat{\delta}} = d(b - 2c)/[4b(a - c)]$. The corresponding critical value of the discount factor is $\hat{\delta} = (b-2c)/(2a-b)$. Defining $\delta_1 = 2c/(d-2a)$ and $\delta_2 = b/(d-b)$ it holds that $0 < \delta_1 < \delta_2 < \hat{\delta} < 1$. Mitra and Nishimura (2001) show that the bifurcation diagram of the family (X, h_δ) is given by figure 6.3.

For $\delta \in (\hat{\delta}, 1)$, the fixed point $\hat{x}_\delta$ is globally stable and, consequently, there do not exist any periodic points of period $p \geq 2$. For $\delta \in (\delta_2, \hat{\delta})$, the fixed point $\hat{x}_\delta$ is unstable and there exists $x_\delta^* \in (0, \hat{x}_\delta)$ such that $\{x_\delta^*, 1\}$ is the orbit of a period-2 cycle. All optimal programs starting from $x \neq \hat{x}_\delta$ converge to this period-2 cycle. It holds that $\lim_{\delta \to \delta_2} x_\delta^* = 0$. For $\delta \in (\delta_1, \delta_2)$, the fixed point $\hat{x}_\delta$ is unstable and all optimal programs starting from $x \neq \hat{x}_\delta$ converge to a period-2 cycle with orbit $\{0, 1\}$. Finally, for $\delta \in (0, \delta_1)$, the fixed point $\hat{x}_\delta$ is unstable and there exists $z_\delta^* \in (\hat{x}_\delta, 1)$ such that $\{0, z_\delta^*\}$ is the orbit of a period-2 cycle. All optimal programs starting from $x \neq \hat{x}_\delta$ converge to this period-2 cycle. It holds that $\lim_{\delta \to 0} z_\delta^* = 0$ and $\lim_{\delta \to \delta_1} z_\delta^* = 1$.

6.4 Optimal Chaos

In the early 1980s, the economics profession became aware of the fact that simple economic mechanisms may generate chaotic dynamics; see, e.g., Benhabib and Day (1982) and Day (1982, 1983). It did not take long until it was shown that the occurrence of deterministic chaos does not necessarily rely on market imperfections or on non-standard assumptions. As a matter of fact, the papers by Deneckere and Pelikan (1986) and Boldrin and Montrucchio (1986) demonstrated that chaotic behavior can be optimal in the reduced-form optimal growth model discussed in subsection 6.2.2 above. These results confirmed

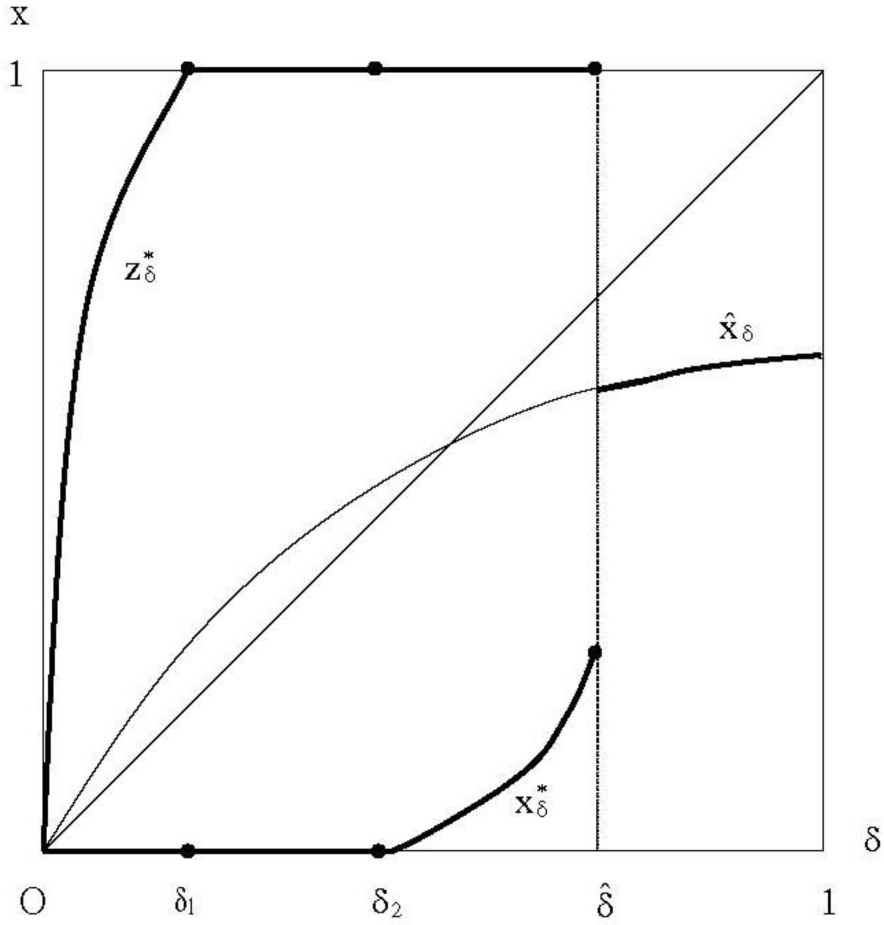

Fig. 6.3. Sutherland's example.

that the standard assumptions of optimal growth theory are logically consistent with endogenously generated business cycles. This insight was very important but it raised also a number of new questions, especially regarding the parameter values for which erratic non-periodic behavior can be optimal.

The only explicit parameter in the reduced form optimal growth model is the discount factor. The research surveyed in chapter 4 of this handbook refined the approach initiated by Boldrin and Montrucchio (1986) and developed discount factor restrictions implied by optimal chaos. Parallel to this development, a number of researchers studied optimal growth models under more detailed structural assumptions on the preferences and the technologies and derived characterizations of the parameter constellations which are consistent with chaos in that framework. By far the most popular framework considered in this literature is the two-sector optimal growth model discussed in subsection 6.2.2 above, but the variations of the Weitzman example already encountered in our discussion of optimal cycles in section 6.3 have also been studied. In the present section we survey the most important contributions to this literature.

6.4.1 Sources of Optimal Chaos

Consider a reduced-form optimal growth model with state space $X = [0, \bar{x}]$ and assume that the transition possibility set takes the form $\Omega = \{(x, z) \mid x \in X,\ 0 \leq z \leq \phi(x)\}$, where $\phi : X \mapsto X$ is a continuous, non-decreasing, and concave function satisfying $\phi(0) = 0$, $\phi(\bar{x}) = \bar{x}$, and $\phi(x) > x$ for all $x \in (0, \bar{x})$. Furthermore, assume that ϕ is continuously differentiable on $(0, \bar{x})$ with derivative ϕ'. It is easy to see that such a transition possibility set can result from a two-sector model with full capital depreciation ($d = 1$) and a production function for the capital good which satisfies $F_x(x, 1) = \phi(x)$ for all $x \in X$. It is also clear that assumption A.1 holds. Furthermore, suppose that assumptions A.2 and A.4 are satisfied. If $U_{12}(x, z)$ is strictly positive for all (x, z) in the interior of Ω, then it follows from corollary 6.3.1(i) that the graph of the optimal policy function h is non-decreasing on X. From the results stated in subsection 6.2.1 we know that all optimal programs must converge to fixed points of (X, h). Thus, chaotic optimal programs are ruled out. A necessary condition for the occurrence of chaotic dynamics in the present situation is therefore that $U_{12}(x, z)$ is negative for some (x, z) in the interior of Ω.

If $U_{12}(x, z) < 0$ holds for all (x, z) in the interior of Ω, then it follows from theorem 6.3.2 that the graph of h is non-increasing whenever it is in the interior of Ω. Nishimura and Yano (1994) elaborate on this observation and describe a method by which one can construct optimal growth models that display topological chaos. The main idea is as follows. First of all, it is assumed that there exists an optimal steady state in the interior of the state space X. Because of the Euler equation, this is tantamount to assuming a solution of the equation $U_2(\hat{x}, \hat{x}) + \delta U_1(\hat{x}, \hat{x}) = 0$ satisfying $0 < \hat{x} < \bar{x}$. Together with the assumption $\phi(x) > x$ for all $x \in (0, \bar{x})$ this implies that the point $(\hat{x}, \hat{x})$ is located on the graph of the optimal policy function and in the interior of Ω. Because the graph of h must be a non-increasing curve whenever it is in the interior of Ω, it follows that there must be $\tilde{x} < \hat{x}$ such that (i) $h(x) = \phi(x)$ for all $x \in [0, \tilde{x}]$ (hence, h is non-decreasing on $[0, \tilde{x}]$), (ii) $h(x) < \phi(x)$ for all $x \in (\tilde{x}, \bar{x}]$, and (iii) h is non-increasing on $[\tilde{x}, \bar{x}]$. The optimal policy function has therefore a tent shape. In a second step one has to make sure that the tent is steep enough in order to generate chaotic dynamics. A sufficient condition for this to be the case is the existence of a period-3 cycle; see Li and Yorke (1975). Nishimura and Yano (1994) construct the period-3 cycle in such a way that two elements of its orbit are in the interval $(0, \tilde{x})$ (that is, they correspond to points on the upper boundary of Ω) while the remaining element is in the interval $(\tilde{x}, \bar{x})$ corresponding to an interior point. Figure 6.4 illustrates the general idea of this construction. The precise conditions under which it is possible are stated in the following theorem in which

$$\begin{aligned}
\Gamma(x,y,z) &= U_2(x,y) + \delta U_1(y,z),\\
\Gamma_{(1)}(x) &= \Gamma(\phi(x), \phi^{(2)}(x), x),\\
\Gamma_{(2)}(x) &= \Gamma(x, \phi(x), \phi^{(2)}(x)) + \delta \Gamma_{(1)}(x)\phi'(\phi(x)),\\
\Gamma_{(3)}(x) &= \Gamma(\phi^{(2)}(x), x, \phi(x)) + \delta \Gamma_{(2)}(x)\phi'(x).
\end{aligned}$$

Theorem 6.4.1. *Consider the optimal growth problem (Ω, U, δ) on $X = [0, \bar{x}]$, where the transition possibility set is $\Omega = \{(x,z) \,|\, x \in X,\ 0 \le z \le \phi(x)\}$ and ϕ is continuous, non-decreasing, and concave on $[0, \bar{x}]$ and continuously differentiable on $(0, \bar{x})$. Suppose that $\phi(0) = 0$, $\phi(\bar{x}) = \bar{x}$, and $\phi(x) > x$ for all $x \in (0, \bar{x})$. Let assumptions A.2-A.4 be satisfied and let h be the optimal policy function. If there exists $x \in (0, \bar{x})$ and $x' \in (0, \bar{x})$ such that $\Gamma_{(1)}(x) \ge 0$, $\Gamma_{(2)}(x) \ge 0$, $\Gamma_{(3)}(x) \le 0$, and $\Gamma_{(3)}(x') > 0$, then it follows that the dynamical system (X, h) exhibits topological chaos.*

Nishimura and Yano (1994) show that the conditions of theorem 6.4.1 can be satisfied in an example, in which the state space is $X = [0,1]$ and the reduced form utility function is given as in the generalized Weitzman example, that is, $U(x,z) = x^\alpha(1-z)^\beta$. As we have seen in subsection 6.3.3 above, this utility function allows for period-2 cycles but it does not allow for more complicated dynamics, if $\Omega = X \times X$. For this reason, Nishimura and Yano (1994) have to choose a non-trivial transition possibility set, that is, they have to specify the function ϕ in such a way that $\phi(x) < 1$ holds for all sufficiently small x.

Theorem 6.4.1 traces the occurrence of optimal chaos to the tent shaped optimal policy function. The tent shape arises because the transition possibility set Ω has a non-trivial and strictly increasing upper boundary and because the optimal policy function is steeply decreasing on the interior of Ω. None of these two properties alone is sufficient to generate optimal chaos, but their combination is. The non-trivial upper boundary of Ω (i.e., the fact that $\phi(x) < 1$ holds for all sufficiently small x) corresponds to the assumption that the economic system cannot move instantaneously from very small states to very large states. This, in turn, can be interpreted as a form of 'upward inertia' of the economic system. The steeply decreasing shape of the optimal policy function in the interior of Ω has two sources: submodularity of U and strong discounting. Submodularity of U means that the maximizer of $U(x,z)$ with respect to z is a decreasing function of x. In other words, if the degree of submodularity is sufficiently strong and the decision maker were myopic ($\delta = 0$), he or she would want to permanently oscillate between small and large states. As the discount factor increases, however, this incentive is increasingly dominated by the decision maker's desire to smooth consumption which follows from the

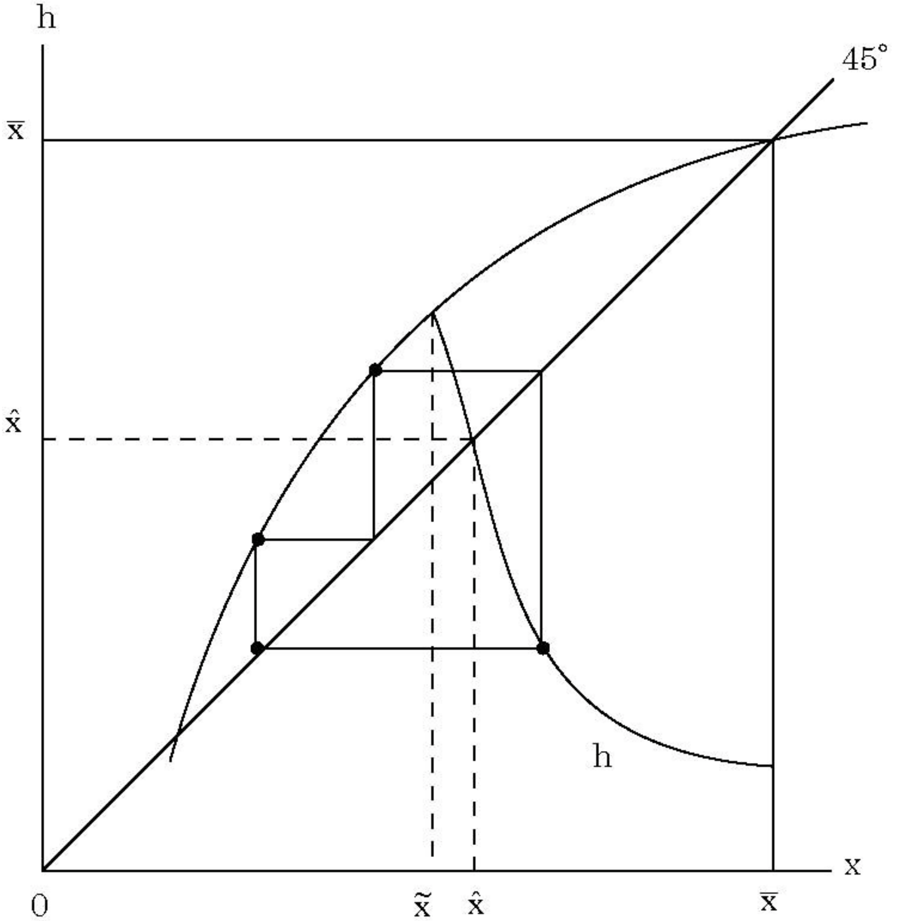

Fig. 6.4. The construction from Nishimura and Yano (1994).

concavity of the utility function. To summarize, Nishimura and Yano (1994) have identified three sources of optimal chaos: upward inertia, submodularity, and strong discounting.

Nishimura and Yano (1995a) replace the assumption of upward inertia of the economic system by 'downward inertia' and show that this can also lead to chaotic optimal programs. They create downward inertia by partial capital depreciation (note that the approach taken by Nishimura and Yano (1994) can be interpreted in the context of the two-sector model provided that capital depreciates fully, i.e., $d = 1$). In the case of partial depreciation the economy cannot move instantaneously from very large states to very small states. Formally, this follows from the fact that the transition possibility set is given by $\Omega = \{(x, z) \,|\, x \in X\,,\ (1-d)x \le z \le \phi(x)\}$. Except for the assumption $d < 1$, the approach taken in Nishimura and Yano (1995a) is the same as in Nishimura and Yano (1994). As before it is assumed that the utility function satisfies A.2 and A.4 as well as $U_{12}(x,z) < 0$ for all (x,z) in the interior of Ω. However, in the present case with less than full depreciation, it follows that there must exist two states $\tilde{x}$ and $\tilde{x}'$ in $(0,\bar{x})$ such that the graph of the optimal policy function h coincides with the upper boundary of Ω for $x \in [0,\tilde{x}]$ and with its lower boundary for $x \in [\tilde{x}',\bar{x}]$. In between the two values $\tilde{x}$ and $\tilde{x}'$, the graph of h is in the interior of Ω and is strictly decreasing. The optimal policy function is therefore not tent-shaped but has an interior maximum $\tilde{x}$ and an interior minimum $\tilde{x}'$. The authors then go on and make the graph of h on the interval $(\tilde{x},\tilde{x}')$ sufficiently steep such that a period-3 cycle exists, which touches the lower boundary of Ω twice whereas the third element of the orbit of the cycle corresponds to a point on the interior section of the optimal policy function. The construction is illustrated in figure 6.5. Conditions very similar to those stated in theorem 6.4.1 are shown to be sufficient for the construction to lead to the desired result. As before, the conditions are shown to be satisfied by a model in which the utility function is given as in the generalized Weitzman example, i.e., $U(x,z) = x^\alpha(1-z)^\beta$. This time, the upper boundary of Ω can be chosen as $\phi(x) = 1$, because the possibility of chaotic dynamics relies on the assumption of downward inertia (partial depreciation $\psi(x) > 0$) rather than on upward inertia ($\phi(x) < 1$).

The paper by Khan and Mitra (2005) also points to downward inertia created by partial depreciation of capital as a possible source of optimal chaos. Kahn and Mitra (2005) consider a discrete-time version of the Robinson-Solow-Srinivasan model with two production sectors. In the notation introduced in subsection 6.2.2, the model is specified by the discount factor $\delta \in (0,1)$, the capital depreciation rate $d \in (0,1)$, and by the functions

$$\begin{aligned} F_c(x_c,\ell_c) &= \min\{x_c,\ell_c\}, \\ F_x(x_x,\ell_x) &= \ell_x/\mu, \\ u(c) &= c. \end{aligned}$$

In other words, the utility function is linear, the production of one unit of the consumption good requires one unit of capital and one unit of labor, and the production of one unit of capital requires μ units of labor (and no capital). The maximal sustainable capital stock is given by $\bar{x} = 1/(d\mu)$ and the transition possibility set is given by $\Omega = \{(x,z) \,|\, 0 \le x \le 1/(d\mu), (1-d)x \le z \le \phi(x)\}$, where $\phi(x) = (1-d)x + (1/\mu)$. Khan and Mitra (2005) first show that, whenever $\delta < \mu$, all optimal programs are described by a continuous optimal policy function h. If in addition to $\delta < \mu$, the parameters μ and d are related in a certain way, then the dynamical system (X,h) is shown to exhibit topological chaos. Finally, Khan and Mitra (2005) prove that, for any value $d \in (0,1)$, there is some μ such that the aforementioned relation between d and μ is indeed satisfied. That is, whenever the rate of depreciation is positive and smaller than

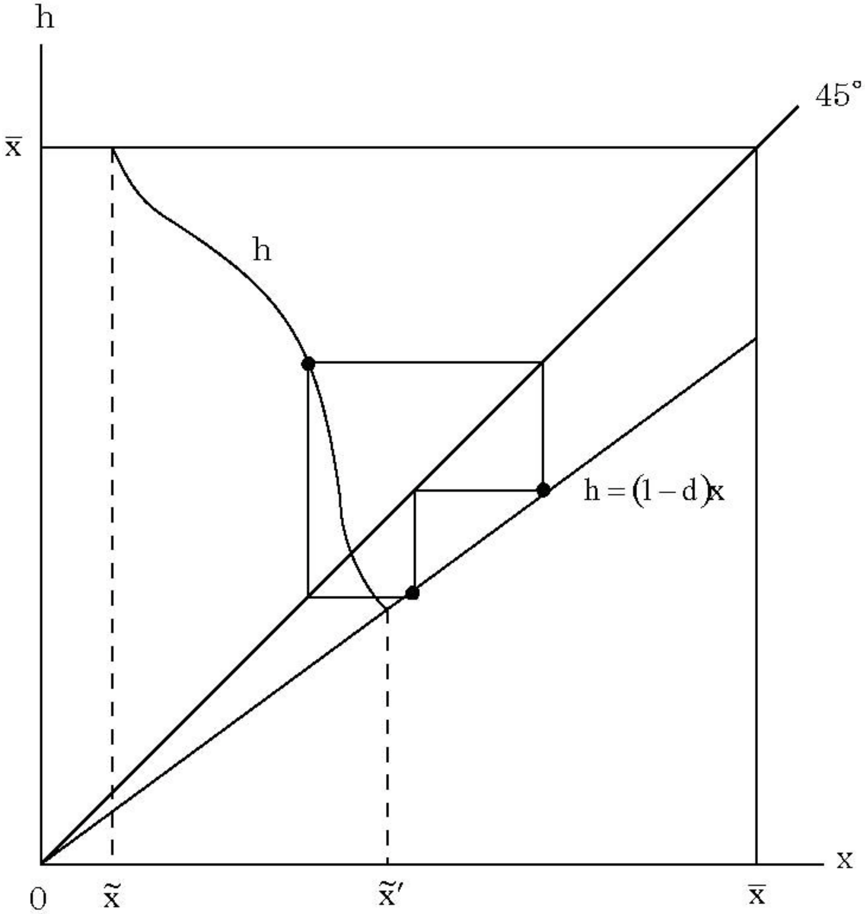

Fig. 6.5. The construction from Nishimura and Yano (1995a).

1, one can find parameters δ and μ such that the model generates topological chaos.

Boldrin and Deneckere (1990) use a combination of analytical and numerical methods to derive interesting insights into the sources of optimal chaos. Their model is a two-sector model with a Cobb-Douglas production function for the consumption good, a Leontief production function for the capital good, and a linear utility function. In the notation of subsection 6.2.2 these assumptions can be written as follows:

$$F_c(x_c, \ell_c) = x_c^\alpha \ell_c^{1-\alpha},$$
$$F_x(x_x, \ell_x) = \mu \min\{x_x, \ell_x/\mu\},$$
$$u(c) = c.$$

This specification gives rise to the transition possibility set $\Omega = \{(x,z) \,|\, 0 \leq x \leq 1, (1-d)x \leq z \leq \phi(x)\}$, where $\phi(x) = \min\{\mu x, 1\}$. The reduced form utility function is

$$U(x,z) = \mu^{-\alpha}[1 + (1-d)x - z]^{1-\alpha}[(1-d+\mu)x - z]^\alpha.$$

As for the parameter values, it is assumed that $\alpha \in (0,1)$ and $\mu > 1/\delta$. The inequality $\mu > 1/\delta$ implies that the marginal product of capital in the investment good sector covers principal and interest in the steady state (recall that the real interest rate in a steady state is $1/\delta - 1$). This assumption guarantees the existence of an interior steady state. The efficient capital-labor ratio in the investment good sector is fixed at $1/\mu$. The factor substitutability in the consumption good sector implies that both factors will be fully employed. Thus, if the economy-wide capital-labor ratio x exceeds $1/\mu$, then it follows that the consumption good sector must be more capital intensive than the investment good sector. Conversely, if $x < 1/\mu$, then consumption goods are produced with lower capital intensity than investment goods. Thus, this model allows for capital intensity reversal.

Boldrin and Deneckere (1990) derive conditions for the existence of cycles of period 2 and 4. Fixing the values of α, μ, and d and treating δ as a bifurcation parameter, they show by means of numerical simulations that successive bifurcations can lead to topological chaos (period-doubling scenario). For example, when $\alpha = 97/100$, $\mu = 100/9$, and $d = 1$, topological chaos is encountered for discount factors between 0.099 and 0.112. They also show that chaos typically disappears rapidly, if one reduces the depreciation rate d from 100% to smaller values, but reappears, if d takes values of 10% or smaller. This suggests that, for almost full depreciation, chaos is generated by the same mechanism as in Nishimura and Yano (1994) (upward inertia and short-run incentives for oscillations), whereas for small values of d it is generated by the mechanism described by Nishimura and Yano (1995a) (downward inertia and short-run incentives for oscillations).

6.4.2 Optimal Chaos Under Weak Impatience

The parametric examples with chaotic optimal policy functions discussed in the papers by Nishimura and Yano (1994, 1995a) and Boldrin and Deneckere (1990) involve the discount factors $\delta = 0.01$, $\delta = 0.05$, and $\delta \approx 0.1$, respectively. These are unrealistically small numbers if the chaotic fluctuations are interpreted as business cycles. The study by Nishimura et al. (1994) shows that chaotic dynamics can occur for all values of the discount factor. In addition, Nishimura et al. (1994) construct optimal policy functions which are not only chaotic in the sense of topological chaos but also in the sense of ergodic chaos and geometric sensitivity. This is accomplished by proving that, for every $\delta \in (0,1)$ and every γ satisfying $1 < \gamma < \min\{2, \delta^{-2}\}$, there exists an optimal growth model (Ω, U, δ) on the state space $X = [0,1]$ with the optimal policy function

$$h(x) = \begin{cases} \gamma x & \text{if } x \in [0, 1/\gamma], \\ 2 - \gamma x & \text{if } x \in [1/\gamma, 1]. \end{cases}$$

Since $\gamma > 1$, it follows from the results by Lasota and Yorke (1973) and Li and Yorke (1978) mentioned in subsection 6.2.1 that this optimal policy function exhibits ergodic chaos and geometric sensitivity. The transition possibility set is chosen to be $\Omega = \{(x,z) \,|\, 0 \le x \le 1, 0 \le z \le \phi(x)\}$ with $\phi(x) = \min\{\gamma x, 1\}$. This shows that the increasing part of the optimal policy function coincides with the boundary of Ω. The decreasing part, however, lies in the interior of Ω. The specification of the utility function U, which is the key step in the construction, is based on ideas developed in Sorger (1992). It leads to an optimal value function which is a simple quadratic polynomial. Thus, Nishimura et al. (1994) have proved the following result.

Theorem 6.4.2. *For every $\delta \in (0,1)$ there exists a reduced form optimal growth model which satisfies assumptions A.1 and A.2, has a strictly concave utility function, and has an optimal policy function exhibiting ergodic chaos and geometric sensitivity.*

Nishimura et al. (1994) also demonstrate that the reduced form optimal growth models in theorem 6.4.2 can be thought of as arising from two-sector models in which both production functions are of the Leontief type and in which the utility function reflects a wealth effect.

Nishimura and Yano (1995b) consider the two-sector growth model with Leontief production functions in both sectors and a linear utility function without any wealth effect. They also demonstrate that ergodic chaos and geometric sensitivity can occur for arbitrary small values of the discount rate. The linear structure imposed by the utility function and the production technologies makes it possible to consider the model as a dynamic linear programming problem; see Nishimura and Yano (1996).

In the notation of subsection 6.2.2, Nishimura and Yano (1995b) assume $d = 1$ and

$$\begin{aligned} F_c(x_c, \ell_c) &= \min\{x_c, \ell_c/\alpha\}, \\ F_x(x_x, \ell_x) &= \mu \min\{x_x, \ell_x/\beta\}, \\ u(c) &= c. \end{aligned}$$

This specification implies that the maximal capital stock that can be reached from x within a single period is given by $\phi(x) = \mu \min\{x, 1/\beta\}$. Therefore, the maximal sustainable capital stock is μ/β and it suffices to restrict attention to the state space $X = [0, \mu/\beta]$. The transition possibility set is given by

$\Omega = \{(x, z) \,|\, 0 \leq x \leq \mu/\beta, \, 0 \leq z \leq \phi(x)\}$. This two-sector model is fully determined by the technological parameters α, β, and μ and the discount factor δ. We shall refer to the model by $M(\alpha, \beta, \mu, \delta)$. The main result from Nishimura and Yano (1995b) can now be stated as follows.

Theorem 6.4.3. *For every $\delta' \in (0, 1)$ there exist parameters α, β, and μ as as well as an open interval $I \subseteq (\delta', 1)$ such that for all $\delta \in I$, the two-sector model $M(\alpha, \beta, \mu, \delta)$ has the optimal policy function h specified by*

$$h(x) = \begin{cases} \mu x & \text{if } x \leq 1/\beta, \\ \mu(1 - \alpha x)/(\beta - \alpha) & \text{if } x \geq 1/\beta, \end{cases} \tag{6.4}$$

and the dynamical system (X, h) exhibits ergodic chaos and geometric sensitivity.

The proof of this result is very technical and will not be presented here. Instead, we will restrict ourselves to a discussion of the main steps of the proof. To begin with, Nishimura and Yano (1995b) assume that the following parameter restrictions are satisfied:

$$\begin{aligned} &\mu > 1/\delta, \\ &\beta > \alpha > 0, \\ &(\beta/\alpha) - 1 < \mu < \beta/\alpha. \end{aligned} \tag{6.5}$$

The condition $\mu > 1/\delta$ ensures the existence of an interior steady state. The inequality $\beta > \alpha > 0$ says that the consumption good sector is more capital intensive than the investment good sector. To explain the meaning of the last parameter restriction in (6.5), it is useful to compute the reduced form utility function

$$U(x, z) = \max\{F_c(x_c, \ell_c) \,|\, x_c + x_x \leq x, \ell_c + \ell_x \leq 1, F_x(x_x, \ell_x) \geq z\}.$$

In the optimal solution to this program, the constraint $\ell_c + \ell_x \leq 1$ is not binding, if $z < f(x)$, and the constraint $x_c + x_x \leq x$ is not binding, if $z > f(x)$. Here the function $f : X \mapsto \mathbb{R}$ is defined by $f(x) = \mu(1 - \alpha x)/(\beta - \alpha)$. Because of the assumption $\beta > \alpha > 0$, it follows that f is a strictly decreasing and continuous function. Consequently, its inverse f^{-1} exists and the condition $z < f(x)$ can also be written as $x < f^{-1}(z)$. In other words, in the optimal solution, labor is not fully employed, if the available capital stock is so small that capital input forms a bottleneck. If the available capital stock exceeds $f^{-1}(z)$, on the other hand, then capital is not fully employed. Full employment of both input factors occurs, if and only if $z = f(x)$.

Note that $f(1/\beta) = \phi(1/\beta) = \mu/\beta$. This shows that the graph of f intersects the upper boundary of the transition possibility set in its kink at $x = 1/\beta$. Nishimura and Yano (1995b) want to ensure that the graph of f does not intersect the lower boundary of Ω and that its slope is smaller than -1. This is exactly what the last line of (6.5) achieves.

Simple calculations show that the reduced form utility function is given by

$$U(x,z) = \begin{cases} x - z/\mu & \text{if } x \le f(z), \\ (\mu - \beta z)/(\alpha\mu) & \text{if } x \ge f(z). \end{cases}$$

It is easy to see from this expression that the indifference curves of U are obtained by translating the curve $z = \phi(x)$ parallel in the direction of the line $z = f(x)$. Nishimura and Yano (1995b) argue that, as long as $x < 1/\beta$, it is not possible to employ the entire labor supply because $z < f(x)$ holds for all $(x,z) \in \Omega$ satisfying $x < 1/\beta$ (capital is the bottleneck). In this case it will therefore be optimal to produce as much capital as possible in order to clear the bottleneck. Once the capital stock x has become larger than $1/\beta$, it is possible to fully employ both factors. The optimal activity in this case will be characterized by full employment of both factors, that is, it will be described by a point on the graph of f. This suggests that the optimal policy function is given by (6.4). From the first and the third line in (6.5) it follows that h is a piecewise linear map with a slope (wherever it is defined) that is larger than 1 in absolute value. As we have seen in subsection 6.2.1, these properties ensure that the dynamical system (X,h) exhibits ergodic chaos and geometric sensitivity.

The crux of the proof in Nishimura and Yano (1995b) consists in showing that h is actually the optimal policy function of $M(\alpha,\beta,\mu,\delta)$. This is a non-trivial issue because the reduced form utility function is not strictly concave in any of its arguments. Nishimura and Yano (1995b) address this problem in two steps. First, they prove that a sufficient condition for h to be the optimal policy function is that (i) the critical point of h, that is $x = 1/\beta$, is a periodic point of (X,h) with period $p > 1$ and (ii) the periodic trajectory from $x = 1/\beta$ is the unique optimal program from $1/\beta$. Second, they show that this sufficient condition can be satisfied for an open interval of discount factors arbitrarily close to 1 provided one lets the period p approach $+\infty$ in an appropriate way.

A somewhat undesirable feature of the result in theorem 6.4.3 is that the optimal policy is non-interior. As a matter of fact, the increasing section of the optimal policy function coincides with the upper boundary of the transition possibility set which implies that, whenever $x \le 1/\beta$, optimal consumption is equal to 0. Note that the same property holds also for the models constructed in the proof of theorem 6.4.2 (as well as for the models discussed in Nishimura and Yano (1994, 1995a)). One may therefore wonder whether chaos can be optimal under weak discounting also in models for which the graph of the optimal policy function is in the interior of the transition possibility set (except at the boundary of the state space X). Nishimura et al. (1998) prove that this is indeed the case. However, whereas Nishimura and Yano (1995b) were able to determine the optimal policy function analytically, in Nishimura et al. (1998) an analytical expression of the optimal policy function is not available and the proof of the existence of optimal chaos is based on a continuity argument.

They consider the model $M_\lambda(\alpha,\beta,\mu,\delta)$ defined by $d = 1$ (full depreciation) and

$$F_c(x_c, \ell_c) = \left[(1/2)x_c^{-1/\lambda} + (1/2)(\ell_c/\alpha)^{-1/\lambda}\right]^{-\lambda},$$
$$F_x(x_x, \ell_x) = \mu_\lambda \left[(1/2)x_x^{-1/\lambda} + (1/2)(\ell_x/\beta)^{-1/\lambda}\right]^{-\lambda},$$
$$u(c) = c^{1-\lambda}/(1-\lambda).$$

Here $\lambda \in (0,1)$ and $\mu_\lambda = 2^{-\lambda}(1+\mu^{1/\lambda})$. It is straightforward to see that, as λ approaches 0, the functions F_c, F_x, and u converge to the corresponding functions used to define $M(\alpha,\beta,\mu,\delta)$. It is also quite obvious that, for $\lambda > 0$, the marginal utility at $c = 0$ is infinitely large and, hence, that consumption along any optimal program starting in $x > 0$ must be strictly positive. Finally, because the production functions are concave and the utility function is strictly concave, the reduced form utility function must be strictly concave with respect to its second argument. This implies that all optimal programs of $M_\lambda(\alpha,\beta,\mu,\delta)$ are described by an optimal policy function. Let us denote this function by h_λ.

Nishimura et al. (1998) now show that, as λ approaches 0, the optimal policy function h_λ converges uniformly to the optimal policy function of $M(\alpha,\beta,\mu,\delta)$, that is, to the function h defined in (6.4). Furthermore, they appeal to a result by Butler and Pianigiani (1978), which implies that a small perturbation of h preserves the existence of topological chaos. Thus, the final conclusion derived by Nishimura et al. (1998) is the following theorem.

Theorem 6.4.4. *Consider the two-sector optimal growth model* $M_\lambda(\alpha,\beta,\mu,\delta)$ *defined above. For every* $\delta' \in (0,1)$ *there exist parameter values* $(\alpha,\beta,\mu,\delta,\lambda)$ *such that* $\delta \in (\delta',1)$ *and such that the optimal policy function of* $M_\lambda(\alpha,\beta,\mu,\delta)$ *exhibits topological chaos.*

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